The Mathematics Of Stock Option Valuation - Part One
An Introduction To The Valuation Process

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When valuing the common stock of a company the appraiser discounts the expected cash flow to the holders of the common stock by a discount rate that includes the risk-free rate and a premium for risk. A call option on that stock is a derivative asset in that it derives its value from the underlying stock. Taking the expected call option payoff and discounting that by a risk-adjusted discount rate almost always leads to an incorrect value. If standard discounting does not apply to stock option valuation then what pricing methodology does? We will discover that whereas the company stock can be viewed as a stand-alone asset (standard discounting applies) the call option on that stock must be viewed as part of a no-arbitrage portfolio (standard discounting does not apply).

Our goal is to explain the Black-Scholes model and how it works. Most texts on the subject make the mistake of introducing the more complex mathematics late in the process such that the reader is lulled into a sense of complacency with the simple algebra of the discrete time model and then blindsided by the dauntingly complex mathematics of the continuous time model. We will take a more logical tact in that we will introduce partial differential equations and risk-neutral pricing earlier on in the process such that the transition to continuous time will be much kinder and even understandable.

We will tackle option pricing in five parts. We will develop the basics using the simple discrete time economy below and then transition to the world of continuous time where the Black-Scholes model reigns. The five parts to this option pricing exercise are...

Part I	An introduction to the valuation process
Part II	Valuing a one-period call option via partial differential equations
Part III	Valuing a one-period call option via risk-neutral probabilities
Part IV	Deriving the continuous-time Black-Scholes equation via partial differential equations
Part V	Deriving the continuous-time Black-Scholes equation via risk-neutral probabilities

To quote the Chinese philosopher Lao-tzu, "A journey of a thousand miles begins with a single step". With that said, let's begin...

The One Period Economy

Our economy has two states of the world at time $t = 1$. In state w_u the stock price moves up to \$180 and in state w_d the stock price moves down to \$80. We have a call option on this stock with an exercise price of \$120 that can be exercised at period $t = 1$. If the stock price is above the exercise price the option will be exercised, otherwise it will be allowed to expire unexercised. The table below presents our one period economy and the two states of the world at time $t = 1$...

Table 1: The One Period Economy

State of the world	w_u	w_d
Stock price	\$180	\$80
Call price	\$60	\$0
Risk-free rate	0.05	0.05
Probability	0.50	0.50

We currently sit at $t = 0$ where the state-of-the-world at $t = 1$ is unknown. We are tasked with placing a value on both the stock and the call at $t = 0$.

Legend of Symbols

S_u	=	Stock price at $t = 1$ given state w_u
S_d	=	Stock price at $t = 1$ given state w_d
S_t	=	Stock price at time t
C_u	=	Call price at $t = 1$ given state w_u
C_d	=	Call price at $t = 1$ given state w_d
C_t	=	Call price at time t
B_t	=	Risk-free zero-coupon bond price at time t
w_u	=	State of the world at $t = 1$ where stock price equals S_u and call price equals C_u
w_d	=	State of the world at $t = 1$ where stock price equals S_d and call price equals C_d
r_f	=	Risk-free rate
θ_s	=	Risk premium applicable to the stock
θ_c	=	Risk premium applicable to the call
t	=	Time period in years

Pricing the Stock

The expected stock price at $t = 1$ using Table 1 above is...

$$\begin{aligned}\mathbb{E}[S_1] &= 180 \times 0.50 + 80 \times 0.50 \\ &= 130 \end{aligned} \tag{1}$$

We determine that the discount rate (k) applicable to this stock is...

$$\begin{aligned} k &= r_f + \theta_s \\ &= 0.05 + 0.25 \\ &= 0.30 \end{aligned} \tag{2}$$

The stock price at $t = 0$ is the discounted value of the expected stock price at $t = 1$. The stock can be valued as a stand-alone asset and therefore standard risk-adjusted discount rates apply. The stock price at $t = 0$ is...

$$\begin{aligned} S_0 &= \mathbb{E}[S_1] \times (1 + k)^{-1} \\ &= 130 \times 1.30^{-1} \\ &= 100 \end{aligned} \tag{3}$$

Pricing the Call

The expected call price at $t = 1$ using Table 1 above is...

$$\begin{aligned}\mathbb{E}[C_1] &= 60 \times 0.50 + 0 \times 0.50 \\ &= 30 \end{aligned} \tag{4}$$

What is the call price at $t = 0$? One possibility is to discount the expected call payoff at $t = 1$ using a discount rate that is comprised of the risk-free rate plus a risk premium. It will not take the appraiser long to come to the conclusion that the value determined this way is not tenable. The problem is that since a call option is a derivative asset (i.e. derives its value from the underlying stock) the call option payoffs can be replicated by a combination of a risk-free bond and shares of the underlying stock. If the discount rate is too high then the call option is too cheap such that an investor can make a positive return at no risk by buying the call and selling the replicating portfolio. If the discount rate is too low then the call option is too expensive such that the investor can make a positive return at no risk by selling the call and buying the replicating portfolio. Unless the appraiser chooses *the right* discount rate by pure chance the call option value determined by standard risk-adjusted discount rates will be bid up (discount rate is too high) or bid down (discount rate is too low) to a value that prevents arbitrage.

We will define arbitrage as the ability to earn a positive return at no risk on a zero investment. An arbitrage portfolio has the following characteristics...

1 Has zero cost to set up
2 Has non-negative values in the future
3 May be of positive value in the future

By creating such a portfolio an investor would receive at no cost the possibility of receiving money in the future.

Scenario One - The Discount Rate is Too High

To illustrate what happens when the discount rate is too high we will choose an arbitrarily high discount rate. The discount rate chosen for this scenario is...

$$\begin{aligned} k &= r_f + \theta_c \\ &= 0.05 + 1.95 \\ &= 2.00 \end{aligned} \tag{5}$$

The estimated value of the call option at $t = 0$ is therefore...

$$\begin{aligned} C_0 &= \mathbb{E}[C_1] \times (1+k)^{-1} \\ &= 30.00 \times 3.00^{-1} \\ &= 10.00 \end{aligned} \tag{6}$$

We will now set up an arbitrage portfolio to take advantage of the mispriced call. To comply with arbitrage portfolio parameter 1 we will require that the portfolio be of zero cost to set up. Since the call is too cheap we will buy the call and sell the replicating portfolio. The transaction at $t = 0$ will be to buy one call, buy a zero-coupon risk-free bond and sell N shares of the underlying stock. The cost to set up the arbitrage portfolio is...

$$NS_0 - B_0 - C_0 = 0 \tag{7}$$

Since we are buying a call that is too cheap we have the opportunity to receive money at no risk in state w_u. To comply with arbitrage portfolio parameters 2 and 3 we want to set up our portfolio such that portfolio value will be greater than zero in state w_u and equal to zero in state w_d. The arbitrage portfolio values at $t = 1$ in all possible states of the world are...

$$w_u: \quad C_u + B_0(1+r_f) - NS_u > 0 \tag{8}$$

$$w_d: \quad C_d + B_0(1+r_f) - NS_d = 0 \tag{9}$$

The first step is to solve for B_0 in equation (9) above...

$$\begin{aligned} C_d + B_0(1+r_f) - NS_d &= 0 \\ B_0 &= (NS_d - C_d)(1+r_f)^{-1} \\ B_0 &= 80N \times 1.05^{-1} \\ B_0 &= 76.19N \end{aligned} \tag{10}$$

The next step is to use the results from equations (6) and (10) above and solve for N in equation (7) above...

$$\begin{aligned} NS_0 - B_0 - C_0 &= 0 \\ 100N - 76.19N - 10.00 &= 0 \\ 23.81N &= 10.00 \\ N &= 0.42 \end{aligned} \tag{11}$$

We know that we have constructed an abribtrage portfolio because the cost to set up the portfolio is zero...

$$\begin{aligned} NS_0 - B_0 - C_0 &= 0 \\ (0.42)(100) - (0.42)(76.19) - 10.00 &= 0 \\ 42.00 - 32.00 - 10.00 &= 0 \end{aligned} \tag{12}$$

Portfolio value is greater than zero at $t = 1$ given state w_u...

$$
\begin{aligned}
C_u + B_0(1 + r_f) - NS_u &> 0 \\
60.00 + (0.42)(76.19)(1.05) - (0.42)(180) &> 0 \\
60.00 + 33.60 - 75.60 &> 0 \\
18.00 &> 0
\end{aligned} \tag{13}
$$

And portfolio value is equal to zero at $t = 1$ given state w_d...

$$
\begin{aligned}
C_d + B_0(1 + r_f) - NS_d &= 0 \\
0 + (0.42)(76.19)(1.05) - (0.42)(80) &= 0 \\
33.60 - 33.60 &= 0
\end{aligned} \tag{14}
$$

Conclusion - A call price of \$10.00 is untenable in that this price will be bid up to a price that prevents arbitrage.

Scenario Two - The Discount Rate is Too Low

To illustrate what happens when the discount rate is too low we will choose the discount rate used to value the stock. The discount rate chosen for this scenario is...

$$
\begin{aligned}
k &= r_f + \theta_c \\
&= 0.05 + 0.25 \\
&= 0.30
\end{aligned} \tag{15}
$$

The estimated value of the call option at $t = 0$ is therefore...

$$
\begin{aligned}
C_0 &= \mathbb{E}[C_1] \times (1 + k)^{-1} \\
&= 30.00 \times 1.30^{-1} \\
&= 23.08
\end{aligned} \tag{16}
$$

We will now set up an arbitrage portfolio to take advantage of the mispriced call. To comply with arbitrage portfolio parameter 1 we will require that the portfolio be of zero value to set up. Since the call is too expensive we will sell the call and buy the replicating portfolio. The transaction at $t = 0$ will be to sell one call, sell a zero-coupon risk-free bond and buy N shares of the underlying stock. The cost to set up the arbitrage portfolio is...

$$
C_0 + B_0 - NS_0 = 0 \tag{17}
$$

Since we are selling a call that is too expensive we have the opportunity to receive money and at no risk in state w_d. To comply with arbitrage portfolio parameters 2 and 3 we want to set up our portfolio such that portfolio value will be equal to zero in state w_u and greater than zero in state w_d. The arbitrage portfolio values at $t = 1$ in all possible states of the world are...

$$
w_u : \quad NS_u - C_u - B_0(1 + r_f) = 0 \tag{18}
$$

$$
w_d : \quad NS_d - C_d - B_0(1 + r_f) > 0 \tag{19}
$$

The first step is to solve for B_0 in equation (18) above...

$$
\begin{aligned}
NS_u - C_u - B_0(1 + r_f) &= 0 \\
B_0 &= (NS_u - C_u)(1 + r_f)^{-1} \\
B_0 &= (180N - 60)(1.05)^{-1} \\
B_0 &= 171.43N - 57.14
\end{aligned} \tag{20}
$$

The next step is to use the results from equations (16) and (20) above and solve for N in equation (17) above...

$$
\begin{aligned}
C_0 + B_0 - NS_0 &= 0 \\
23.08 + 171.43N - 57.14 - 100N &= 0 \\
71.43N &= 34.06 \\
N &= 0.477
\end{aligned} \tag{21}
$$

We know that we have constructed an abribtrage portfolio because the cost to set up the portfolio is zero...

$$\begin{aligned} C_0 + B_0 - NS_0 &= 0 \\ 23.08 + (0.477)(171.43) - 57.14 - (0.477)(100.00) &= 0 \\ 23.08 + 81.76 - 57.14 - 47.70 &= 0 \end{aligned} \tag{22}$$

Portfolio value is zero at time $t = 1$ given state-of-the-world w_u...

$$\begin{aligned} NS_u - C_u - B_0(1 + r_f) &= 0 \\ (0.477)(180) - 60.00 - ((0.477)(171.43) - 57.14)(1.05) &= 0 \\ 85.86 - 60.00 - 25.86 &= 0 \end{aligned} \tag{23}$$

And portfolio value is greater than zero at time $t = 1$ given state-of-the-world w_d...

$$\begin{aligned} NS_d - C_d - B_0(1 + r_f) &> 0 \\ (0.477)(80) - 0 - ((0.477)(171.43) - 57.14)(1.05) &> 0 \\ 38.16 - 25.86 &> 0 \\ 12.30 &> 0 \end{aligned} \tag{24}$$

Conclusion - A call price of \$23.08 is untenable in that this price will be bid down up to a price that prevents arbitrage.

Conclusions

We have seen from the above scenarios that if the appraiser values the call as a stand-alone asset the call value will most likely be incorrect. If the discount rate is too high then the call value is too low such that the call price is bid up to a price that prevents arbitrage. If the discount rate is too low then the call value is too high such that the call price is bid down to a price that prevents arbitrage. The appraiser can solve this dilema by calculating the call's no-arbitrage price directly. To do this the appraiser must view the call as part of a no-arbitrage portfolio rather than as a stand-alone asset.

The no-arbitrage price of a derivative product such as a call can be determined via (1) partial differential equations, which is introduced in Part II, or (2) risk-neutral pricing, which is introduced in Part III. To derive the Black-Scholes option pricing model equations Black and Scholes used the former approach although use of the latter approach results in identical equations.

Now that we have covered the basics we can move on to Part II which introduces the reader to partial differential equations as a method to determine the no-arbitrage price of the call.